International Institute of Information Technology, Hyderabad

(Deemed to be University)

MA4.101-Real Analysis (Monsoon-2025)

Assignment 2

Due date: **December 2, 2025** Total Marks: 80

Some definitions and facts.

Adherent points. Let $X \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be an adherent point of X if for all $\varepsilon > 0$ there exists $y \in X$ such that $|y - x| \leq \varepsilon$. Equivalently, $(x - \varepsilon, x + \varepsilon) \cap X \neq \emptyset$ for all $\varepsilon > 0$.

Limit points. Let $X \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be a limit point of X if for all $\varepsilon > 0$ there exists $y \in X \setminus \{x\}$ such that $|y - x| \leq \varepsilon$. Equivalently, $(x - \varepsilon, x + \varepsilon) \cap (X \setminus \{x\}) \neq \emptyset$ for all $\varepsilon > 0$.

Closure and closed sets. Let $X \subseteq \mathbb{R}$. The closure $\overline{X}$ of X may be defined as either: (1) the set of all adherent points of X, or (2) the union of X with its limit points. A set X is said to be closed if $\overline{X} = X$.

Continuity of functions. Let $X \subseteq \mathbb{R}$, let $x_0 \in X$, and let $f : X \to \mathbb{R}$ be a function. The function f is said to be continuous at x_0 if either of the following equivalent conditions holds: (1) For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$ with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \varepsilon$. (2) For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$ with $|x - x_0| \leq \delta$ we have $|f(x) - f(x_0)| \leq \varepsilon$. Thus the use of $<$ or $\leq$ in these definitions is irrelevant.

Derivative of functions. Let $X \subseteq \mathbb{R}$, let x_0 be a limit point of X, and let $f : (0, \infty) \to \mathbb{R}$ be a function defined as $f(x) = x^\alpha$, where $\alpha \in \mathbb{R}$. Then the derivative of f at $x_0 \in (0, \infty)$ is given by $f'(x_0) = \alpha\, x_0^{\alpha-1}$.

Question 1 [10 Marks]. Let $L \in \mathbb{R}$. Let (x_n) be a real sequence and let $T_n := x_{n+3} - 3x_{n+2} + 3x_{n+1} - x_n$ and $s_n := \frac{x_{n+2}+x_n}{2}$. If

$$\text{(P1)} \lim_{n\to\infty} T_n = 0, \text{ and (P2)} \lim_{n\to\infty} s_n = L.$$

Prove that (x_n) is convergent.

Question 2 [10 Marks]. Let $L \in \mathbb{R}$. Let $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ be real sequences satisfying properties.

$$\text{(P1)} \quad x_n \leq y_n \ \forall\, n \geq 0, \qquad \text{(P2)} \quad \lim_{n\to\infty} (x_n + y_n) = 2L,$$
$$\text{(P3)} \quad \lim_{n\to\infty} (y_{n+1} - x_n) = 0, \qquad \text{(P4)} \quad \lim_{n\to\infty} (y_n - x_{n+1}) = 0.$$

Prove that both (x_n) and (y_n) converge, and that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = L$. [Hint/Caution: In this problem, the existence of $\lim x_n$ and $\lim y_n$ is *what we must prove*, so limit algebra such as $\lim(y_{n+1} - x_n) = \lim y_{n+1} - \lim x_n$ is not applicable. Try writing a recurrence relation for $(y_{n+1} - x_{n+1}) := t_{n+1}$ involving convergent sequences from question.]

Question 3 [10 Marks]. Let the real numbers a and b satisfy $a < b$. Let the function $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Define

$$L(t) = f(a) + \frac{f(b) - f(a)}{b - a}(t - a), \qquad t \in [a,b],$$

and define $K(t) = f(t) - L(t),\ t \in [a,b]$.

(A). **[3 Marks]** Show that K satisfies the hypotheses of Rolle's theorem on $[a,b]$.

(B). **[7 Marks]** Define the function $\Phi(t) = K(t) - K(a+b-t),\ t \in [a,b]$ and show that there exists a point $c \in (a,b)$ for which

$$f'(c) + f'(a+b-c) = 2\frac{f(b) - f(a)}{b - a}.$$

Question 4 [10 Marks]. Let $M > 0$ and $a < b$. Let $f : [a,b] \to \mathbb{R}$ be a function which is continuous on $[a,b]$ and differentiable on (a,b), and such that

$$|f'(x)| \leq M \qquad \text{for all } x \in (a,b).$$

Show that for any $x, y \in [a, b]$ one has

$$|f(x) - f(y)| \leq M\,|x - y|.$$

[Hint: Apply the Mean Value Theorem to f on a suitable subinterval. Functions satisfying the inequality

$$|f(x) - f(y)| \leq M|x - y|$$

are called *Lipschitz continuous* with Lipschitz constant M. Thus functions with bounded derivative are Lipschitz continuous.]

Question 5 [10 Marks]. Let $X \subset \mathbb{R}$ and define the distance from a point $x \in \mathbb{R}$ to X by

$$d(x, X) := \inf\{|x - y| : y \in X\}.$$

(A) [**4 Marks**] Show that a point $x \in \mathbb{R}$ belongs to the closure $\overline{X}$ of X if and only if

$$d(x, X) = 0.$$

(B) [**4 Marks**] Let $Y \subset \mathbb{R}$ be another subset. Show that for all $x \in \mathbb{R}$:

$$d(x, X \cup Y) = \min\{d(x, X), d(x, Y)\}, \qquad d(x, X \cap Y) \geq \max\{d(x, X), d(x, Y)\}.$$

(C) [**2 Marks**] Show that X is closed if and only if for every $x \in \mathbb{R}$,

$$d(x, X) = 0 \implies x \in X.$$

Question 6 [10 Marks]. Let X and Y be arbitrary subsets of $\mathbb{R}$. Let $\overline{X}$ and $\overline{Y}$ be the closures of X and Y in $\mathbb{R}$, respectively. Prove the following.

(A). [**2 Marks**] $X \subseteq \overline{X}$.

(B). [**2 Marks**] $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$.

(C). [**2 Marks**] $\overline{X \cap Y} \subseteq \overline{X} \cap \overline{Y}$.

(D). [**2 Marks**] If $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$.

(E). [**2 Marks**] $\overline{(\overline{X} \cap \overline{Y})} = \overline{X} \cap \overline{Y}$.

Question 7 [10 Marks]. Let $(a_n)_{n\geq 1}$ be a real sequence, and $c_n := \frac{a_n+2a_{n+1}}{3}$ for each $n \geq 1$. Assume that the following three series converge:

$$O := \sum_{n=1}^{\infty} a_{2n-1}, \qquad C := \sum_{n=1}^{\infty} c_n, \qquad C_{\text{odd}} := \sum_{n=1}^{\infty} c_{2n-1}.$$

Prove that the full series $\sum_{n=1}^{\infty} a_n$ converges and equals $(O + 3\,C_{\text{odd}})/2$.

Question 8 [10 Marks]. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

(A). [**5 marks**] For which points $x_0 \in \mathbb{R}$ is f continuous? Prove your answer.

(B). [**5 marks**] For which points $x_0 \in \mathbb{R}$ is f differentiable? Prove your answer and compute the derivative where it exists.